

THE ASYMPTOTIC GROWTH OF GRADED LINEAR SERIES ON ARBITRARY PROJECTIVE SCHEMES

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ABSTRACT. Recently, Okounkov [16], Lazarsfeld and Mustata [13], and Kaveh and Khovanskii [10] have shown that the growth of a graded linear series on a projective variety over an algebraically closed field is asymptotic to a polynomial. We give a complete description of the possible asymptotic growth of graded linear series on projective schemes over a perfect field. If the scheme is reduced, then the growth is polynomial like, but the growth can be very complex on nonreduced schemes.

We also give an example of a graded family of m -primary ideals $\{I_n\}$ in a nonreduced d -dimensional local ring R , such that the length of R/I_n divided by n^d does not have a limit, even when restricted to any arithmetic sequence.

1. INTRODUCTION

In this paper we investigate the asymptotic growth of a graded linear series on an arbitrary projective scheme over a perfect field. Recent results of Okounkov [16], Lazarsfeld and Mustata [13], and Kaveh and Khovanskii [10] show the remarkable fact that on a projective variety over an algebraically closed field k , the dimension $\dim_k L_n$ of a graded linear series L is asymptotic for large n to the value of a polynomial; that is,

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{mn}}{n^q}$$

exists, where m is the index of L and $q \geq 0$ is the Kodaira-Iitaka dimension of L . We recall this result in Theorem 3.1 below, which is the general statement of Kaveh and Khovanskii [10]. The limit can be irrational, even when L is the section ring of a big line bundle, as is shown by Srinivas and the author in Example 4 of Section 7 [4].

It is natural to consider the question of the existence of such limits when we loosen these conditions. We give a complete description of how much of Theorem 3.1 extends and how much does not extend to arbitrary projective schemes over a perfect field. In Theorem 4.4 we show that the exact statement of Theorem 3.1 holds for graded linear series on a projective variety over a perfect field. In Theorem 5.2 we show that the theorem generalizes very well to graded linear series on a reduced projective variety over a perfect field, although the statement requires a slight modification. The conclusion is that there is a positive integer r such that for any integer a , the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^q}$$

exists whenever n is constrained to lie in the arithmetic sequence $a + br$. Here q is the Kodaira-Iitaka dimension of L . A nontrivial example on a connected, reduced, equidimensional but not irreducible projective scheme is given in Example 5.5.

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This is however the extent to which Theorem 3.1 generalizes. We give a series of examples of graded linear series on non reduced projective schemes where such limits do not exist.

The most significant example is Example 6.3, which is of a “big” graded linear system L with maximal Kodaira-Iitaka dimension d on a nonreduced but irreducible d -dimensional projective scheme such that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^d}$$

does not exist, even when n is constrained to lie in *any* arithmetic sequence.

We give a related example, Example 7.1, showing the failure of limits of lengths of quotients of a graded family of m_R -primary ideals $\{I_n\}$ in the d -dimensional local ring $R = k[[x_1, \dots, x_d, y]]/y^2$. In the example, the limit

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

does not exist, even when n is constrained to lie in *any* arithmetic sequence.

Such limits for graded families of ideals in local rings are shown to exist in many cases by work of Ein, Lazarsfeld and Smith [6], Mustata [15], Lazarsfeld and Mustata [13] and of the author [3]. In Theorem 5.9 of [3], we show that the limit (1) always exists when $\{I_n\}$ is a graded family of m_R -primary ideals in a d -dimensional analytically unramified equicharacteristic local ring with perfect residue field, and give a number of applications of this result. It follows from Example 7.1 that the assumption of analytically unramified cannot be removed from Theorem 5.9 [3].

A fundamental property of the Kodaira-Iitaka dimension $q(L)$ for a graded linear series L on a reduced projective scheme with $q(L) \geq 0$ is that there exists a constant β such that there is an upper bound

$$\dim_k L_n < \beta n^{q(L)}$$

for all n . This is a classical result for complete linear systems on varieties, and is part of the foundations of the Kodaira-Iitaka dimension (Theorem 10.2 [9]). A proof of this inequality for graded linear series on reduced projective varieties over perfect fields is given in this paper in Corollary 5.3. However, this equality may fail on non reduced projective schemes. We always have an upper bound

$$\dim_k L_n < \gamma n^d$$

where d is the dimension of the scheme (4), but it is possible on a d -dimensional nonreduced projective scheme to have $q(L) = -\infty$ and have growth of order n^d . We give a simple example where this happens in Example 6.4. In fact, it is quite easy to construct badly behaved examples with $q(L) = -\infty$, since in this case the condition that $L_m L_n \subset L_{m+n}$ required for a graded linear series may be vacuous.

We give in Example 6.6 an example of a section ring of a line bundle \mathcal{N} on a non reduced but irreducible d -dimensional projective scheme Z with growth of order n^{d-1} such that for any positive integer r , there exists an integer a such that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(Z, \mathcal{N}^n)}{n^{d-1}}$$

does not exist when n is constrained to lie in the arithmetic sequence $a + br$.

Even on nonreduced projective schemes, we do have the classical property of Kodaira-Iitaka dimension that if $q(L) \geq 0$, then there is a positive constant α and a positive integer

m such that

$$\alpha n^{q(L)} < \dim_k L_{mn}$$

for all integers n (5).

The volume of a line bundle \mathcal{L} on a d -dimensional variety X is the limsup

$$(2) \quad \text{Vol}(\mathcal{L}) = \limsup_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!}.$$

There has been spectacular progress of our understanding of the volume as a function on the big cone in $N^1(X)$ on a projective variety X over an algebraically closed field (where (2) is actually a limit). Much of the theory is explained in [12], where extensive references are given. Volume is continuous on $N^1(X)$ but is not twice differentiable on all of $N^1(X)$ (as shown in an example of Ein Lazarsfeld, Mustata, Nakamaye and Popa, [5]). Bouksom, Favre and Jonsson [1] have shown that the volume is \mathcal{C}^1 -differentiable on the big cone of $N^1(X)$. Interpretation of the directional derivative in terms of intersection products and many interesting applications are given in [1], [5] and [13].

The starting point of the theory of volume on nonreduced schemes is to determine if the limsup defined in (2) exists as a limit. We see from Examples 6.3 and 6.4 that the limit does not always exist for graded linear series L . However, neither of these examples are section rings of a line bundle. The examples are on the nonreduced scheme X which is a double linear hyperplane in a projective space \mathbb{P}^{d+1} . All line bundles on X are restrictions of line bundles on \mathbb{P}^{d+1} , so that if \mathcal{L} is a line bundle on X , then $h^0(X, \mathcal{L}^n)$ is actually a polynomial in n , and $\text{Vol}(\mathcal{L})$ not only exists as a limit, it is even a rational number.

We essentially use the notation of Hartshorne [8]. For instance, a variety is required to be integral. We will denote the maximal ideal of a local ring R by m_R . If ν is a valuation of a field K , then we will write V_ν for the valuation ring of ν , and m_ν for the maximal ideal of V_ν . We will write Γ_ν for the value group of ν . If A and B are local rings, we will say that B dominates A if $A \subset B$ and $m_B \cap A = m_A$.

2. GRADED LINEAR SERIES AND THE KODAIRA-IITAKA DIMENSION

Suppose that X is a d -dimensional projective scheme over a field k , and \mathcal{L} is a line bundle on X . Then under the natural inclusion of rings $k \subset \Gamma(X, \mathcal{O}_X)$, we have that the section ring

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$$

is a graded k -algebra. A graded k -subalgebra $L = \bigoplus_{n \geq 0} L_n$ of a section ring of a line bundle \mathcal{L} on X is called a *graded linear series* for \mathcal{L} .

We define the *Kodaira-Iitaka dimension* $q = q(L)$ of a graded linear series L as follows. Let

$$\sigma(L) = \max \left\{ m \mid \begin{array}{l} \text{there exists } y_1, \dots, y_m \in L \text{ which are homogeneous of positive} \\ \text{degree and are algebraically independent over } k \end{array} \right\}.$$

$q(L)$ is then defined as

$$q(L) = \begin{cases} \sigma(L) - 1 & \text{if } \sigma(L) > 0 \\ -\infty & \text{if } \sigma(L) = 0 \end{cases}$$

This definition is in agreement with the classical definition for line bundles on projective varieties (Definition in Section 10.1 [9]). We give a summary of a few formulas which hold for the Kodaira-Iitaka dimension on general projective schemes. We defer proofs of these formulas to the appendix at the end of this paper.

Lemma 2.1. *Suppose that L is a graded linear series on a d -dimensional projective scheme X over a field k . Then*

1.

$$(3) \quad q(L) \leq d = \dim X.$$

2. *There exists a positive constant γ such that*

$$(4) \quad \dim_k L_n < \gamma n^d$$

for all n .

3. *Suppose that $q(L) \geq 0$. Then there exists a positive constant α and a positive integer e such that*

$$(5) \quad \dim_k L_{en} > \alpha n^{q(L)}$$

for all positive integers n .

4. *Suppose that X is reduced and L is a graded linear series on X . Then $q(L) = -\infty$ if and only if $L_n = 0$ for all $n > 0$.*

3. SOME REMARKABLE LIMITS

Suppose that L is a graded linear series on a projective variety X . The *index* $m = m(L)$ of L is defined as the index of groups

$$m = [\mathbb{Z} : G]$$

where G is the subgroup of \mathbb{Z} generated by $\{n \mid L_n \neq 0\}$.

Theorem 3.1. *(Okounkov [16], Lazarsfeld and Mustata [13], Kaveh and Khovanskii [10]) Suppose that X is a projective variety over an algebraically closed field k , and L is a graded linear series on X . Let $m = m(L)$ be the index of L and $q = q(L) \geq 0$ be the Kodaira-Iitaka dimension of L . Then*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^q}$$

exists.

In particular, from the definition of the index, we have that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^q}$$

exists, whenever n is constrained to lie in an arithmetic sequence $a + bm$ ($m = m(L)$ and a an arbitrary but fixed constant).

It follows that $\dim_k L_n = 0$ if $m \nmid n$, and if $q(L) \geq 0$, then there exist positive constants $\alpha < \beta$ such that

$$(6) \quad \alpha n^q < \dim_k L_{nm} < \beta n^q$$

for all sufficiently large positive integers n

The proof of the theorem is by an ingenious method, reducing the problem to computing the volume of a section (the Newton-Okounkov body) of an appropriate cone.

Corollary 3.2. *(Okounkov [16], Lazarsfeld and Mustata [13]) Suppose that X is a projective variety of dimension d over an algebraically closed field k , and \mathcal{L} is a big line bundle on X (the Kodaira-Iitaka dimension of the section ring of \mathcal{L} is d). Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d}$$

exists.

This corollary was earlier proven using Fujita approximation, [7], Example 11.4.7 [12], [17].

An example of a big line bundle where the limit in Theorem 3.1 and Corollary 3.2 is an irrational number is given in Example 4 of Section 7 [4].

4. LIMITS ON VARIETIES OVER NON CLOSED FIELDS

The proof of Koveh and Khovanskii of Theorem 3.1 actually shows the following.

Theorem 4.1. (*Koveh and Khovanskii*) Suppose that X is a d -dimensional projective variety over an arbitrary field k and there exists a valuation ν of the function field $k(X)$ of X which vanishes on nonzero elements of k , has a value group Γ_ν which is isomorphic as a group to \mathbb{Z}^d , and the residue field of V_ν/m_ν is k . Then the conclusions of Theorem 3.1 are valid for any graded linear series L on X with $q(L) \geq 0$.

These conditions on the valuation are stated before Definition 2.26 [10]. The condition that ν has “one dimensional leaves” is defined before Proposition 2.6 [10]. It is equivalent to the condition that the residue field of the valuation is k .

Proof. Suppose that such a valuation ν exists. Let V_ν be the valuation ring of ν in the function field $k(X)$. Let Q be the center of ν on X ; that is, $\mathcal{O}_{X,Q}$ is the local ring of X which is dominated by V_ν . Q exists and is unique since a projective variety is proper. Q is a k -rational point on X since the residue field of V_ν is k .

L is a graded linear series for some line bundle \mathcal{L} on X . Let $m = m(L)$ and $q = q(L)$. Since X is integral, \mathcal{L} is isomorphic to an invertible sheaf $\mathcal{O}_X(D)$ for some Cartier divisor D on X . We can assume that Q is not contained in the support of D , after possibly replacing D with a Cartier divisor linearly equivalent to D . We have an induced graded k -algebra isomorphism of section rings

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \rightarrow \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$$

which takes L to a graded linear series for $\mathcal{O}_X(D)$. Thus we may assume that $\mathcal{L} = \mathcal{O}_X(D)$. For all n , the restriction map followed by inclusion into V_ν ,

$$(7) \quad \Gamma(X, \mathcal{L}^n) \rightarrow \mathcal{L}_Q^n = \mathcal{O}_{X,Q}(nD) \subset V_\nu$$

is a 1-1 k -vector space homomorphism since X is integral, and we have an induced k -algebra homomorphism

$$L \rightarrow \mathcal{O}_{X,Q} \subset V_\nu.$$

Given a nonnegative element γ in the value group Γ_ν of ν , which is isomorphic to \mathbb{Z}^d as a group, with some total ordering, we have associated valuation ideals I_γ and I_γ^+ in V_ν defined by

$$I_\gamma = \{f \in V_\nu \mid \nu(f) \geq \gamma\}$$

and

$$I_\gamma^+ = \{f \in V_\nu \mid \nu(f) > \gamma\}.$$

Since $V_\nu/m_\nu = k$, we have the critical condition that

$$(8) \quad \dim_k I_\gamma / I_\gamma^+ = 1$$

for all non negative $\gamma \in \Gamma_\nu$. Let

$$S_n = \{\gamma \in \Gamma_\nu \mid \text{there exists } f \in L_n \text{ such that } \nu(f) = \gamma\}.$$

By (8) and (7), we that

$$\dim_k L_n \cap I_\gamma / L_n \cap I_\gamma^+ = \begin{cases} 1 & \text{if there exists } f \in L_n \text{ with } \nu(f) = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Since every element of L_n has non negative value (as $L_n \subset V_\nu$), we have that

$$(9) \quad \dim_k L_n = |S_n|$$

for all n . Let

$$S(L) = \{(\gamma, n) | \gamma \in S_n\},$$

a subsemigroup of \mathbb{Z}^{d+1} . By Theorem 2.30 [10], $S(L)$ is a “strongly nonnegative semi-group”, so by Theorem 1.26 [10] and (9), we have that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^q} = \lim_{n \rightarrow \infty} \frac{|S_{mn}|}{n^q}$$

exists, where $m = m(L)$, and is proportional to the volume of the Newton-Okounkov body $\Delta(S(L)) = \pi^{-1}(m) \cap \text{Con}(S(L))$, where $\text{Con}(S(L))$ is the closure of the cone generated by $S(L)$ in \mathbb{R}^{d+1} and $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ is the projection onto the last factor. □

The condition that there exists a valuation as in the assumptions of Theorem 4.1 is always satisfied if k is algebraically closed. It is however a rather special condition over non closed fields, as is shown by the following proposition.

Proposition 4.2. *Suppose that X is a d -dimensional projective variety over a field k . Then there exists a valuation ν of the function field $k(X)$ of X such that the value group Γ_ν of ν is isomorphic to \mathbb{Z}^d and the residue field $V_\nu/m_\nu = k$ if and only if there exists a birational morphism $X' \rightarrow X$ of projective varieties such that there exists a nonsingular (regular) k -rational point $Q' \in X'$.*

Proof. First suppose there exists a valuation ν of the function field $k(X)$ of X such that the value group Γ_ν of ν is isomorphic to \mathbb{Z}^d as a group and with residue field $V_\nu/m_\nu = k$. Then ν is an “Abhyankar valuation”; that is

$$\text{trdeg}_k k(X) = d = 0 + d = \text{trdeg}_k V_\nu/m_\nu + \text{rational rank } \Gamma_\nu,$$

with $k = V_\nu/m_\nu$, so there exists a local uniformization of ν by [11]. Let Q be the center of ν on X , so that V_ν dominates $\mathcal{O}_{X,Q}$. $\mathcal{O}_{X,Q}$ is a localization of a k -algebra $k[Z]$ where $Z \subset V_\nu$ is a finite set. By Theorem 1.1 [11], there exists a regular local ring R which is essentially of finite type over k with quotient field $k(X)$ such that V_ν dominates R and $Z \subset R$. Since $k[Z] \subset R$ and V_ν dominates $\mathcal{O}_{X,Q}$, we have that R dominates $\mathcal{O}_{X,Q}$. The residue field $R/m_R = k$ since V_ν dominates R . There exists a projective k -variety X'' such that R is the local ring of a closed k -rational point Q' on X'' , and the birational map $X'' \dashrightarrow X$ is a morphism in a neighborhood of Q' . Let X' be the graph of the birational correspondence between X'' and X . Since $X'' \dashrightarrow X$ is a morphism in a neighborhood of Q' , the projection of X' onto X'' is an isomorphism in a neighborhood of Q' . We can thus identify Q' with a nonsingular k -rational point of X' .

Now suppose that there exists a birational morphism $X' \rightarrow X$ of projective varieties such that there exists a nonsingular k -rational point $Q' \in X'$.

Choose a regular system of parameters y_1, \dots, y_d in $R = \mathcal{O}_{X',Q'}$. $R/m_R = k(Q') = k$, so k is a coefficient field of R . We have that $\hat{R} = k[[y_1, \dots, y_d]]$. We define a valuation $\hat{\nu}$

dominating \hat{R} by stipulating that

$$(10) \quad \hat{\nu}(y_i) = e_i \text{ for } 1 \leq i \leq d$$

where $\{e_i\}$ is the standard basis of the totally ordered group $(\mathbb{Z}^d)_{\text{lex}}$, and $\hat{\nu}(c) = 0$ if c is a nonzero element of k .

If $f \in \hat{R}$ and $f = \sum c_{i_1, \dots, i_d} y_1^{i_1} \cdots y_d^{i_d}$ with $c_{i_1, \dots, i_d} \in k$, then

$$\hat{\nu}(f) = \min\{\nu(y_1^{i_1} \cdots y_d^{i_d}) \mid c_{i_1, \dots, i_d} \neq 0\}.$$

We let ν be the valuation of the function field $k(X)$ which is obtained by restricting ν . The value group of ν is $(\mathbb{Z}^d)_{\text{lex}}$.

Suppose that h is in $k(X)$ and $\nu(h) = 0$. Write $h = \frac{f}{g}$ where $f, g \in R$ and $\nu(f) = \nu(g)$. Thus in \hat{R} , we have expansions $f = \alpha y_1^{i_1} \cdots y_d^{i_d} + f'$, $g = \beta y_1^{i_1} \cdots y_d^{i_d} + g'$ where α, β are nonzero elements of k , $\nu(y_1^{i_1} \cdots y_d^{i_d}) = \nu(f) = \nu(g)$ and $\nu(f') > \nu(f)$, $\nu(g') > \nu(g)$. Let $\gamma = \frac{\alpha}{\beta}$ in k . Computing $f - \gamma g$ in \hat{R} , we obtain that $\nu(f - \gamma g) > \nu(f)$, and thus the residue of $\frac{f}{g}$ in V_ν/m_ν is equal to the residue of γ , which is in k . By our construction $k \subset V_\nu$. Thus the residue field $V_\nu/m_\nu = k$. \square

Corollary 4.3. *Suppose that X is a projective variety over a field k which has a nonsingular k -rational point. Then the conclusions of Theorem 3.1 hold for any graded linear series L on X .*

Proof. This is immediate from Theorem 4.1 and Proposition 4.2. \square

We obtain the following extension of Theorem 3.1.

Theorem 4.4. *Suppose that X is a projective variety over a perfect field k .*

Let L be a graded linear series on X . Let $m = m(L)$ be the index of L and $q = q(L) \geq 0$ be the Kodaira-Iitaka dimension of L . Then

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^q}$$

exists.

Proof. Let Q be a closed regular point in X . Let $R = \mathcal{O}_{X,Q}$. Let k' be a Galois closure of the residue field $k(Q)$ of R over k . k' is finite separable over k , so that $X' = X \times_k k'$ is reduced as $\mathcal{O}_X \otimes_k k'$ is a subsheaf of rings of $k(X) \otimes_k k'$, which is reduced by Theorem 39, Section 15, Chapter III [18].

Let $S = R \otimes_k k'$. Then S is a reduced semi local ring by Theorem 39 [18]. Let p_1, \dots, p_r be the maximal ideals of S . $S/m_R S \cong (R/m_R) \otimes_k k'$ is reduced by Theorem 39, [18]. Thus $m_R S = p_1 \cap \cdots \cap p_r$. Since R is a regular local ring, m_R is generated by $d = \dim R$ elements. For $1 \leq i \leq r$, we thus have that $p_i S_{p_i} = m_R S_{p_i}$ is generated by $d = \dim R = \dim S_{p_i}$ elements. Thus S_{p_i} is a regular local ring for all i , so S is a regular ring.

$k' = k[\alpha]$ for some $\alpha \in k'$ since k' is a finite separable extension of k . Let $f(x) \in k[x]$ be the minimal polynomial of α . k' is a separable normal extension of k containing α , so $f(x)$ splits into distinct linear factors in $k'[x]$. Then

$$\bigoplus_{i=1}^r S/p_i \cong S/m_R S \cong k' \otimes_k k' \cong k'[x]/(f(x)) \cong (k')^r.$$

Thus $S/p_i \cong k'$ for all i . Let Q'_i be the corresponding closed point to p_i in X' , which has the local ring $\mathcal{O}_{X', Q'_i} = S_{p_i}$, so that Q'_i is a regular, k' -rational point on the variety X' for all i .

Let X_1, \dots, X_s be the distinct irreducible components of X' . Since X' is reduced, we have a natural inclusion

$$0 \rightarrow \mathcal{O}_{X'} \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{X_i}$$

which induces inclusions

$$(11) \quad \Gamma(X', \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}) \rightarrow \bigoplus_{i=1}^s \Gamma(X_i, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i})$$

for all n .

The elements of the Galois group G of k' over k induce X -automorphisms of X' which act transitively on the components X_i . G acts naturally on $\mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$. Thus for $\sigma \in G$, we have a commutative diagram

$$\begin{array}{ccccc} L_n \otimes_k k' & \subset & \Gamma(X', \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}) & \rightarrow & \Gamma(X_1, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_1}) \\ \downarrow \text{id} & & \downarrow \sigma & & \downarrow \sigma \\ L_n \otimes_k k' & \subset & \Gamma(X', \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}) & \rightarrow & \Gamma(\sigma(X_1), \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{\sigma(X_1)}). \end{array}$$

Suppose that $h \in L_n \otimes_k k'$ maps to zero in $\Gamma(X_1, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_1})$. Since G acts transitively on the components of X' , h maps to zero in $\Gamma(X_i, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i})$ for all i . From the inclusion (11), we conclude that $h = 0$. Thus we have inclusions

$$L_n \otimes_k k' \rightarrow \Gamma(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_1})$$

for all n .

Let $L' = \bigoplus_{n \geq 0} L_n \otimes_k k'$. L' is a graded linear series for the line bundle $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_1}$ on the k' -variety \bar{X}_1 . We have that $m(L') = m(L)$ and $q(L') = q(L)$.

Since $Q_i \in X_1$ for some i , we have that X_1 contains a non singular k' -rational point. By Corollary 4.3, the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_{k'} L'_{nm}}{n^q}$$

thus exists.

Now the theorem follows from the formula

$$\dim_k L_n = \dim_{k'} L_n \otimes_k k'.$$

□

It follows from the theorem that (6) holds for a graded linear series on a projective variety over a perfect field k .

We obtain that Corollary 3.2 holds on reduced projective schemes over perfect fields.

Corollary 4.5. *Suppose that X is a reduced projective scheme of dimension d over a perfect field k , and \mathcal{L} is a line bundle on X . Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d}$$

exists.

Proof. We first prove the corollary in the case when X is integral (a variety). We may assume that the section ring L of \mathcal{L} has maximal Kodaira-Iitaka dimension d , because the limit is zero otherwise. There then exists a positive constant α and a positive integer e such that

$$\dim_k \Gamma(X, \mathcal{L}^{ne}) > \alpha n^d$$

for all positive integers n by (5). Let H be a hyperplane section of X , giving a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0.$$

Tensoring with \mathcal{L}^n and taking global sections, we see that $\Gamma(X, \mathcal{L}^{ne} \otimes \mathcal{O}_X(-H)) \neq 0$ for $n \gg 0$ as $q(\mathcal{L}^e \otimes \mathcal{O}_H) \leq \dim(H) = d - 1$. Since H is ample, there exists a positive integer f such that $\mathcal{L} \otimes \mathcal{O}_X(fH)$ is generated by global sections. Thus

$$\Gamma(X, \mathcal{L}^{nef+1}) \cong \Gamma(X, (\mathcal{L}^{nef} \otimes \mathcal{O}_X(-fH)) \otimes (\mathcal{L} \otimes \mathcal{O}_X(fH))) \neq 0$$

for $n \gg 0$. Thus $m(L) = 1$. The corollary in the case when X is a variety thus follows from Theorem 4.4.

Now assume that X is only reduced. Let X_1, \dots, X_s be the irreducible components of X . Since X is reduced, we have a natural short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{n \geq 0} \mathcal{O}_{X_i} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} has support of dimension $\leq d - 1$. Tensoring with \mathcal{L}^n , we obtain that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d} = \sum_{i=1}^s \lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X_i, \mathcal{L}^n \otimes \mathcal{O}_{X_i})}{n^d}$$

exists, as $\dim_k \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ grows at most like n^{d-1} . \square

5. LIMITS ON REDUCED SCHEMES

Suppose that X is a projective scheme over a field k and L is a graded linear series for a linebundle \mathcal{L} on X . Suppose that Y is a closed subscheme of X . Set $\mathcal{L}|Y = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Taking global sections of the natural surjections

$$\mathcal{L}^n \xrightarrow{\varphi_n} (\mathcal{L}|Y)^n \rightarrow 0,$$

for $n \geq 1$ we have induced short exact sequences of k -vector spaces

$$(12) \quad 0 \rightarrow K(L, Y)_n \rightarrow L_n \rightarrow (L|Y)_n \rightarrow 0,$$

where

$$(L|Y)_n := \varphi_n(L_n) \subset \Gamma(Y, (\mathcal{L}|Y)^n)$$

and $K(L, Y)_n$ is the kernel of $\varphi_n|L_n$. Defining $K(L, U)_0 = k$ and $(L|Y) + 0 = \varphi_0(L_0)$, we have that $L|Y = \bigoplus_{n \geq 0} (L|Y)_n$ is a graded linear series for $\mathcal{L}|Y$ and $K(L, Y) = \bigoplus_{n \geq 0} K(L, Y)_n$ is a graded linear series for \mathcal{L} .

Lemma 5.1. *Suppose that X is a reduced projective scheme and X_1, \dots, X_s are the irreducible components of X . Suppose that L is a graded linear series on X . Then*

$$q(L) = \max\{q(L|X_i) \mid 1 \leq i \leq s\}.$$

Proof. L is a graded linear series for a line bundle \mathcal{L} on X . Let X_1, \dots, X_s be the irreducible components of X . Since X is reduced, we have a natural inclusion

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{X_i}.$$

There is a natural inclusion of k -algebras

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \rightarrow \bigoplus_{i=1}^s \left(\bigoplus_{n \geq 0} \Gamma(X_i, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}) \right),$$

which induces an inclusion of k -algebras

$$(13) \quad L \rightarrow \bigoplus_{i=1}^s L|X_i.$$

Suppose that i is such that $1 \leq i \leq s$. Set $t = q(L|X_i)$. Then by the definition of Kodaira-Itaka dimension, there exists a graded inclusion of k -algebras $\varphi : k[z_1, \dots, z_t] \rightarrow L|X_i$ where $k[z_1, \dots, z_t]$ is a graded polynomial ring. Since the projection $L \rightarrow L|X_i$ is a surjection, we have a lift of φ to a graded k -algebra homomorphism into L , which is 1-1, so that $q(L) \geq t$. Thus

$$q(L) \geq \max\{q(L|X_i) \mid 1 \leq i \leq s\}.$$

Let $q = q(L)$. Then there exists a 1-1 k -algebra homomorphism $\varphi : k[z_1, \dots, z_q] \rightarrow L$ where $k[z_1, \dots, z_q]$ is a positively graded polynomial ring. Let $\varphi_i : k[z_1, \dots, z_q] \rightarrow L|X_i$ be the induced homomorphisms, for $1 \leq i \leq s$. Let \mathfrak{p}_i be the kernel of φ_i . Since (13) is 1-1, we have that $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s = (0)$. Since $k[z_1, \dots, z_q]$ is a domain, this implies that some $\mathfrak{p}_i = (0)$. Thus φ_i is 1-1 and we have that $q(L|X_i) \geq q(L)$. \square

Theorem 5.2. *Suppose that X is a reduced projective scheme over a perfect field k . Let L be a graded linear series on X . Let $q = q(L) \geq 0$ be the Kodaira-Itaka dimension of L . Then there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^q}$$

exists for any fixed $a \in \mathbb{N}$.

The theorem says that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^q}$$

exists if n is constrained to lie in an arithmetic sequence $a + br$ with r as above, and for some fixed a . The conclusions of the theorem are a little weaker than the conclusions of Theorem 4.4 for integral varieties. In particular, the index $m(L)$ has no relevance on reduced but not irreducible varieties.

Proof. Let X_1, \dots, X_s be the irreducible components of X . Define graded linear series M^i on X by $M^0 = L$, $M^i = K(M^{i-1}, X_i)$ for $1 \leq i \leq s$. By (12), for $n \geq 1$, we have exact sequence of k -vector spaces

$$0 \rightarrow (M^{j+1})_n = K(M^j, X_{j+1})_n \rightarrow M_n^j \rightarrow (M^j|X_{j+1})_n \rightarrow 0$$

for $1 \leq j \leq s-1$, and

$$M_n^j = \text{Kernel}(L_n \rightarrow \bigoplus_{i=1}^s (L|X_i)_n)$$

for $0 \leq j \leq s$. As in (13) in the proof of Lemma 5.1, $L \rightarrow \bigoplus_{i=1}^s L|X_i$ is an injection of k -algebras since X is reduced. Thus $M_n^s = (0)$, and

$$(14) \quad \dim_k L_n = \sum_{i=1}^s \dim_k (M^{i-1}|X_i)_n$$

for all n . Let $r = \text{LCM}\{m(L|X_i) \mid q(L|X_i) = q(L)\}$. The theorem now follows from Theorem 4.4 applied to each of the X_i with $q(M^{i-1}|X_i) = q(L)$ (we can start with X_1 with $q(L|X_1) = Q(L)$). \square

Corollary 5.3. *Suppose that X is a reduced projective scheme over a perfect field k . Let L be a graded linear series on X with $q(L) \geq 0$. Then there exists a positive constant β such that*

$$(15) \quad \dim_k L_n < \beta n^{q(L)}$$

for all n . Further, there exists a positive constant α and a positive integer m such that

$$(16) \quad \alpha n^{q(L)} < \dim_k L_{mn}$$

for all positive integers n .

Proof. Equation (15) follows from (14), since $\dim_k(M^{i-1}|X_i) \leq \dim_k(L|X_i)$ for all i , and since (6) holds on a variety. Equation (16) is immediate from (5). \square

The following lemma is required for the construction of the next example. It follows from Theorem V.2.17 [8] when $r = 1$. The lemma uses the notation of [8].

Lemma 5.4. *Let k be an algebraically closed field, and write $\mathbb{P}^1 = \mathbb{P}_k^1$. Suppose that $r \geq 0$. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r)$ with natural projection $\pi : X \rightarrow \mathbb{P}^1$. Then the complete linear system $|\Gamma(X, \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1))|$ is base point free, and the only curve contracted by the induced morphism of X is the curve C which is the section of π defined by the projection of $\mathcal{O}(-1)_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^r$ onto the first factor.*

Proof. We prove this by induction on r .

First suppose that $r = 0$. Then π is an isomorphism, and $X = C$.

$$\mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \pi_*\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1},$$

from which the statement of the lemma follows.

Now suppose that $r > 0$ and the statement of the lemma is true for $r - 1$. Let V_0 be the \mathbb{P}^1 -subbundle of X corresponding to projection onto the first $r - 1$ factors,

$$(17) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}^r \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}^{r-1} \rightarrow 0.$$

Apply π_* to the exact sequence

$$0 \rightarrow \mathcal{O}_X(1) \otimes \mathcal{O}_X(-V_0) \rightarrow \mathcal{O}_X(1) \rightarrow \mathcal{O}_{V_0}(1) \rightarrow 0$$

to obtain the exact sequence (17), from which we see that $\mathcal{O}_X(V_0) \cong \mathcal{O}_X(1)$ and $V_0 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}^{r-1})$ with $\mathcal{O}_X(V_0) \otimes \mathcal{O}_{V_0} \cong \mathcal{O}_{V_0}(1)$. Let F be the fiber over a point in \mathbb{P}^1 by π . We have that $\mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_X(V_0 + F)$. Apply π_* to

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_X(V_0 + F) \rightarrow \mathcal{O}_X(V_0 + F) \otimes \mathcal{O}_{V_0} \rightarrow 0$$

to get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \bigoplus \mathcal{O}_{\mathbb{P}^1}(1)^r \rightarrow \pi_*(\mathcal{O}_{V_0}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow 0.$$

Now take global sections to obtain that the restriction map

$$\Gamma(X, \mathcal{O}_X(V_0 + F)) \rightarrow \Gamma(V_0, \mathcal{O}_{V_0}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1))$$

is a surjection. In particular, by the induction statement, V_0 contains no base points of $\Lambda = |\Gamma(X, \mathcal{O}_X(V_0 + F))|$. Since any two fibers F over points of \mathbb{P}^1 are linearly equivalent, Λ is base point free.

Suppose that γ is a curve of X which is not contained in V_0 . If $\pi(\gamma) = \mathbb{P}^1$ then $(\gamma \cdot F) > 0$ and $(\gamma \cdot V_0) \geq 0$ so that γ is not contracted by Λ . If γ is in a fiber of F then $(\gamma \cdot F) = 0$. Let $F \cong \mathbb{P}^r$ be the fiber of π containing γ . Let $h = F \cdot V_0$, a hyperplane section of F . Then $(\gamma \cdot V_0) = (\gamma \cdot h)_F > 0$. Thus γ is not contracted by Λ . By induction on r , we have that

C is the only curve on V_0 which is contracted by Λ . We have thus proven the induction statement for r . \square

Example 5.5. Let k be an algebraically closed field. Suppose that s is a positive integer and $a_i \in \mathbb{Z}_+$ are positive integers for $1 \leq i \leq s$. Suppose that $d > 1$. Then there exists a connected reduced projective scheme X over k which is equidimensional of dimension d with a line bundle \mathcal{L} on X and a bounded function $\sigma(n)$ such that

$$\dim_k \Gamma(X, \mathcal{L}^n) = \lambda(n) \binom{d+n-1}{d-1} + \sigma(n),$$

where $\lambda(n)$ is the periodic function

$$\lambda(n) = |\{i \mid n \equiv 0(a_i)\}|.$$

The Kodaira-Iitaka dimension of L is $q(L) = d - 1$. Let $m' = \text{LCM}\{a_i\}$. The limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^{d-1}}$$

exists whenever n is constrained to be in an arithmetic sequence $a + bm'$ (with any fixed a). We have that $\dim_k L_n \neq 0$ for all n if some $a_i = 1$, so the conclusions of Theorem 3.1 do not quite hold in this example.

Proof. Let E be an elliptic curve over k . Let p_0, p_1, \dots, p_s be points on E such that the line bundles $\mathcal{O}_E(p_i - p_0)$ have order a_i . Let $S = E \times_k \mathbb{P}_k^{d-1}$, and define line bundles $\mathcal{L}_i = \mathcal{O}_E(p_i - p_0) \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1)$ on S . The Segre embedding gives a closed embedding of S in \mathbb{P}^r with $r = 3d - 1$. Let $\pi : X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r) \rightarrow \mathbb{P}^1$ be the projective bundle, and let C be the section corresponding to the surjection of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r$ onto the first factor. Let b_1, \dots, b_s be distinct points of \mathbb{P}^1 and let F_i be the fiber by π over b_i . Let S_i be an embedding of S in F_i . We can if necessary make a translation of S_i so that the point $c_i = C \cdot F_i$ lies on S_i , but is not contained in $p_j \times \mathbb{P}^{d-1}$ for any j . We have a line bundle \mathcal{L}' on the (disjoint) union T of the S_i defined by $\mathcal{L}'|_{S_i} = \mathcal{L}_i$.

By Lemma 5.4, there is a morphism $\varphi : X \rightarrow Y$ which only contracts the curve C . φ is actually birational and an isomorphism away from C , but we do not need to verify this, as we can certainly obtain this after replacing φ with the Stein factorization of φ . Let $Z = \varphi(T)$. The birational morphism $T \rightarrow Z$ is an isomorphism away from the points c_i , which are not contained on the support of the divisor defining \mathcal{L}' . Thus $\mathcal{L}'|_{(T \setminus \varphi(C))}$ extends naturally to a line bundle \mathcal{L} on Z .

We have a short exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{S_i} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} has finite support. Tensoring this sequence with \mathcal{L}^n and taking global sections, we obtain that

$$0 \leq \sum_{i=1}^s \dim_k \Gamma(S_i, \mathcal{L}_i^n) - \dim_k \Gamma(Z, \mathcal{L}^n) \leq \dim_k \mathcal{F}$$

for all n . Since

$$\Gamma(S_i, \mathcal{L}_i^n) \cong \Gamma(E, \mathcal{O}_E(n(p_i - p_0))) \otimes_k \Gamma(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(n))$$

by the Kuenneth formula, we obtain the conclusions of the example. \square

6. PERVERSITY ON NONREDUCED SCHEMES

In this section we give a series of examples, showing interesting growth of graded linear series on nonreduced projective schemes. The examples show that the wonderful theorems about growth for graded linear series on varieties do not generalize to nonreduced schemes.

6.1. An example with maximal Kodaira-Iitaka dimension. Let $i_1 = 2$ and $r_1 = \frac{i_1}{2}$. For $j \geq 1$, inductively define i_{j+1} so that i_{j+1} is even and $i_{j+1} > 2^j i_j$. Let $r_{j+1} = \frac{i_{j+1}}{2}$. For $n \in \mathbb{Z}_+$, define

$$(18) \quad \sigma(n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{i_j}{2} & \text{if } i_j \leq n < i_{j+1} \end{cases}$$

Lemma 6.1. *Suppose that $a \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Then given $m > 0$ and $\varepsilon > 0$, there exists a positive integer $n = a + br$ with $b \in \mathbb{N}$ such that $n > m$ and*

$$\left| \frac{\sigma(n)}{n} - \frac{1}{2} \right| < \varepsilon$$

Proof. Choose j sufficiently large that $i_j > m$, $i_j + r < i_{j+1}$ and

$$(19) \quad \frac{i_j}{2(i_j + k)} > \frac{1}{2} - \varepsilon$$

for $0 \leq k < r$. There exists $n = i_j + k$ with $0 \leq k < r$ in the arithmetic sequence $a + br$.

$$\frac{\sigma(n)}{n} = \frac{i_j}{2n} = \frac{i_j}{2(i_j + k)}.$$

By (19),

$$\frac{1}{2} \geq \frac{i_j}{2(i_j + k)} > \frac{1}{2} - \varepsilon.$$

□

Lemma 6.2. *Suppose that $a \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Then given $m > 0$ and $\varepsilon > 0$, there exists a positive integer $n = a + br$ with $b \in \mathbb{N}$ such that $n > m$ and*

$$\left| \frac{\sigma(n)}{n} \right| < \varepsilon.$$

Proof. Choose j sufficiently large that $i_j > m + r$, $2^j i_j > r$ and

$$(20) \quad \frac{i_j}{2(2^j i_j - k)} < \varepsilon$$

for $0 < k \leq r$. Let $n = i_{j+1} - k$ with $0 < k \leq r$ in the arithmetic sequence $a + br$.

$$\frac{\sigma(n)}{n} = \frac{i_j}{2n} = \frac{i_j}{2(i_{j+1} - k)}.$$

By (20),

$$0 < \frac{i_j}{2(i_{j+1} - k)} < \varepsilon.$$

□

It follows from the previous two lemmas that

$$\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n}$$

does not exist, even when n is constrained to lie in an arithmetic sequence.

Example 6.3. Let k be a field, and let X be the d -dimensional projective nonreduced k -scheme consisting of a double linear hyperplane in \mathbb{P}_k^{d+1} , with $d \geq 1$. There exists a graded linear series L for $\mathcal{O}_X(2)$ with maximal Kodaira-Iitaka dimension $q(L) = d$, such that

$$(21) \quad \lim_{n \rightarrow \infty} \frac{\dim_k(L_n)}{n^d}$$

does not exist, even when n is constrained to lie in an arithmetic sequence.

L_n is constructed to be a subspace of $\Gamma(X, \mathcal{O}_X(2n))$ which generates $\mathcal{O}_X(2n)$ at all points except along a fixed $d - 1$ dimensional linear subspace of X .

Proof. We can choose homogeneous coordinates on \mathbb{P}_k^2 so that $X = \text{Proj}(S)$ where $S = k[x_0, x_1, \dots, x_{d+1}]/(x_1^2)$, and the fixed linear subspace is $Z(x_0, x_1) \subset X$. Let \bar{x}_i be the classes of x_i in S , so that $S = k[\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{d+1}]$ (with $\bar{x}_1^2 = 0$). Define a graded family of homogeneous ideals in S by

$$J_n = (\bar{x}_0^n, \bar{x}_1 \bar{x}_0^{n-\sigma(n)})$$

for $n \geq 1$ and $J_0 = S$. We have that $J_m J_n \subset J_{m+n}$ since $\sigma(j) \leq \sigma(k)$ for $j \leq k$.

Let \tilde{J}_n be the sheafification of J_n on X . Define

$$L_n = \Gamma(X, \tilde{J}_n \otimes \mathcal{O}_X(2n)) \subset \Gamma(X, \mathcal{O}_X(2n)) = S_{2n}.$$

The L_n define a graded linear series L on X .

Let M_i be the set of all monomials in the $d + 1$ variables $\bar{x}_0, \bar{x}_2, \dots, \bar{x}_{d+1}$ of degree i .

L_n has a k -basis consisting of $\bar{x}_0^n M_{n-\alpha}$ and $\bar{x}_1 \bar{x}_0^{n-\sigma(n)} M_{n+\sigma(n)-1}$. Thus

$$(22) \quad \dim_k L_n = \binom{d+n}{d} + \binom{d+n+\sigma(n)-1}{d}.$$

Let $P(t)$ be the degree d rational polynomial $P(t) = \binom{d+t}{d}$. We have that

$$(23) \quad P(t) = \frac{t^d}{d!} + \text{lower order terms in } t.$$

Let $n = a + br$ be an arithmetic sequence (with a, r fixed). Suppose that n_0 is a positive integer and $\varepsilon > 0$ is a real number.

Since $0 \leq \sigma(n) \leq \frac{n}{2}$ for all n , it follows from (22) and (23) that there exists $n_1 \geq n_0$ such that $n \geq n_1$ implies

$$\left| \frac{\dim_k L_n}{n^d} - \frac{1}{d!} \left(1 + \left(1 + \frac{\sigma(n)}{n} \right)^d \right) \right| < \frac{\varepsilon}{2}.$$

By Lemma 6.1, there exists an integer $n_2 > n_1$ such that n_2 is in the arithmetic sequence $n = a + br$ and

$$\left| \frac{1}{d!} \left(1 + \left(1 + \frac{\sigma(n_2)}{n_2} \right)^d \right) - \frac{1 + (1 + \frac{1}{2})^d}{d!} \right| < \frac{\varepsilon}{2}.$$

Thus

$$(24) \quad \left| \frac{\dim_k L_{n_2}}{n_2^d} - \frac{1 + (\frac{3}{2})^d}{d!} \right| < \varepsilon.$$

On the other hand, by Lemma 6.2, there exists an integer $n_3 > n_1$ such that n_3 is in the arithmetic sequence $n = a + br$ and

$$\left| \frac{1}{d!} \left(1 + \left(1 + \frac{\sigma(n_3)}{n_3} \right)^d \right) - \frac{2}{d!} \right| < \frac{\varepsilon}{2}.$$

Thus

$$(25) \quad \left| \frac{\dim_k L_{n_3}}{n_3^d} - \frac{2}{d!} \right| < \varepsilon.$$

By (24) and (25) we have that the limit (21) does not exist when n is constrained to line in the arithmetic sequence $n = a + br$. Since this sequence was arbitrary, we have obtained the conclusions of the example. \square

The above example is a graded linear series on a double linear hyperplane X in \mathbb{P}^{d+1} . We observe that if \mathcal{L} is a line bundle on X , then not only does the limit exist for the section ring of \mathcal{L} , it is even a rational number. Suppose that \mathcal{L} is a line bundle on X . Let X_0 be the reduced linear subspace of \mathbb{P}^{d+1} which has the same support as X . We have a short exact sequence of coherent \mathcal{O}_{X_0} modules

$$0 \rightarrow \mathcal{O}_{X_0}(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_0} \rightarrow 0.$$

From the exact sequence

$$H^1(X, \mathcal{O}_{X_0}(-1)) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(X_0) \rightarrow H^2(X, \mathcal{O}_{X_0}(-1))$$

of Exercise III.4.6 [8], and from the cohomology of projective space, we see that restriction gives an isomorphism $\text{Pic}(X) \cong \text{Pic}(X_0)$. Since every line bundle on the linear subspace X_0 is the restriction of a line bundle on \mathbb{P}^{d+1} , we have that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^{d+1}}(a) \otimes \mathcal{O}_X$ for some $a \in \mathbb{Z}$. From the exact sequences

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{d+1}}(na - 2) \rightarrow \mathcal{O}_{\mathbb{P}^{d+1}}(na) \rightarrow \mathcal{L}^n \rightarrow 0,$$

we see that

$$\dim_k \Gamma(X, \mathcal{L}^n) = \dim_k \Gamma(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(na)) - \dim_k \Gamma(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(na - 2)),$$

which is zero for all positive n if $a < 0$, is 1 for all positive n if $a = 0$, and is equal to the polynomial

$$\binom{na + (d+1)}{d+1} - \binom{na - 2 + (d+1)}{d+1}$$

for $n \geq 2$ if $a > 0$.

6.2. Examples with Kodaira-Iitaka dimension $-\infty$. It is much easier to construct perverse examples with Kodaira-Iitaka dimension $-\infty$, since the condition $L_m L_n \subset L_{m+n}$ can be trivial in this case. If X is a reduced variety, and L is a graded linear series on X , then it follows from Corollary 5.3 that there is an upper bound $\dim_k L_n < \beta n^{q(L)}$ for all n . However, for nonreduced varieties of dimension d , we only have the upper bound $\dim_k < \gamma n^d$ of (4). Here is an example with $q(L) = -\infty$ and maximal growth of order n^d .

Example 6.4. *Let k be a field, and let X be the one dimensional projective non reduced k -scheme consisting of a double line in \mathbb{P}_k^2 . Let T be a subset of the positive integers. There exists a graded linear series L for $\mathcal{O}_X(2)$ such that*

$$\dim_k L_n = \begin{cases} n+1 & \text{if } n \in T \\ 0 & \text{if } n \notin T \end{cases}$$

In the example, we have that $q(L) = -\infty$, but $\dim_k L_n$ is $O(n) = O(n^{\dim X})$.

Proof. We can choose homogeneous coordinates on \mathbb{P}_k^2 so that $X = \text{Proj}(S)$, where $S = k[x_0, x_1, x_2]/(x_1^2)$. Let \bar{x}_i be the classes of x_i in S , so that $S = k[\bar{x}_0, \bar{x}_1, \bar{x}_2]$. Define a graded linear series L for $\mathcal{O}_X(2)$ by defining L_n to be the k -subspace of $\Gamma(X, \mathcal{O}_X(2n))$ spanned by $\{\bar{x}_1 \bar{x}_0^i \bar{x}_2^j \mid i + j = n\}$ if $n \in T$ and $L_n = 0$ if $n \notin T$. Then

$$\dim_k L_n = \begin{cases} n + 1 & \text{if } n \in T \\ 0 & \text{if } n \notin T \end{cases}$$

□

We modify the above example a little bit to find another example with interesting growth.

Theorem 6.5. *Let k be a field, and let X be the one dimensional projective non reduced k -scheme consisting of a double line in \mathbb{P}_k^2 . Let T be any infinite subset of the positive integers \mathbb{Z}_+ such that $\mathbb{Z}_+ \setminus T$ is also infinite. There exists a graded linear series L for $\mathcal{O}_X(2)$ such that*

$$\dim_k L_n = \begin{cases} \lceil \log(n) \rceil + 1 & \text{if } n \in T \\ \lceil \frac{\log(n)}{2} \rceil + 1 & \text{if } n \notin T \end{cases}$$

In this example we have $q(L) = \infty$.

Proof. We can choose homogeneous coordinates on \mathbb{P}_k^2 so that $X = \text{Proj}(S)$, where $S = k[x_0, x_1, x_2]/(x_1^2)$. Let \bar{x}_i be the classes of x_i in S , so that $S = k[\bar{x}_0, \bar{x}_1, \bar{x}_2]$. Define

$$\lambda(n) = \begin{cases} \lceil \log(n) \rceil & \text{if } n \in T \\ \lceil \frac{\log(n)}{2} \rceil & \text{if } n \in \mathbb{Z}_+ \setminus T. \end{cases}$$

Define a graded linear series L for $\mathcal{O}_X(2)$ by defining L_n to be the k -subspace of $\Gamma(X, \mathcal{O}_X(n))$ spanned by

$$\bar{x}_0^n \bar{x}_1, \bar{x}_0^{n-1} \bar{x}_1 \bar{x}_2, \dots, \bar{x}_0^{n-\lambda(n)} \bar{x}_1 \bar{x}_2^{\lambda(n)}.$$

Then L_n has the desired property. □

The following is an example of a line bundle on a non reduced scheme for which there is interesting growth. The characteristic $p > 0$ plays a role in the construction.

Example 6.6. *Suppose that $d \geq 1$. There exists an irreducible but nonreduced projective variety Z of dimension d over a field of positive characteristic p , and a line bundle \mathcal{N} on Z , whose Kodaira-Iitaka dimension is $-\infty$, such that*

$$\dim_k \Gamma(Z, \mathcal{N}^n) = \begin{cases} \binom{d+n-1}{d-1} & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases}$$

In particular, given a positive integer r , there exists at least one integer a with $0 \leq a < r$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(Z, \mathcal{N}^n)}{n^{d-1}}$$

does not exist when n is constrained to lie in the arithmetic sequence $a + br$.

Proof. Suppose that p is a prime number such that $p \equiv 2 \pmod{3}$. In Section 6 of [4], a projective genus 2 curve C over an algebraic function field k of characteristic p is constructed, which has a k -rational point Q and a degree zero line bundle \mathcal{L} with the properties that

$$\dim_k \Gamma(C, \mathcal{L}^n \otimes \mathcal{O}_C(Q)) = \begin{cases} 1 & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases}$$

and

$$(26) \quad \Gamma(C, \mathcal{L}^n) = 0 \text{ for all } n.$$

Let $\mathcal{E} = \mathcal{O}_C(Q) \oplus \mathcal{O}_C$. Let $S = \mathbb{P}(\mathcal{E})$ with natural projection $\pi : S \rightarrow C$, a ruled surface over C . Let C_0 be the section of π corresponding to the surjection onto the second factor $\mathcal{E} \rightarrow \mathcal{O}_C \rightarrow 0$. By Proposition V.2.6 [8], we have that $\mathcal{O}_S(-C_0) \otimes_{\mathcal{O}_S} \mathcal{O}_{C_0} \cong \mathcal{O}_C(Q)$. Let X be the nonreduced subscheme $2C_0$ of S . We have a short exact sequence

$$0 \rightarrow \mathcal{O}_C(Q) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0.$$

Let $\mathcal{M} = \pi^*(\mathcal{L}) \otimes_{\mathcal{O}_S} \mathcal{O}_X$. Then we have short exact sequences

$$(27) \quad 0 \rightarrow \mathcal{L}^n \otimes_{\mathcal{O}_C} \mathcal{O}_C(Q) \rightarrow \mathcal{M}^n \rightarrow \mathcal{L}^n \rightarrow 0.$$

By (27) and (26), we have that

$$\begin{aligned} \dim_k \Gamma(X, \mathcal{M}^n) &= \dim_k \Gamma(C, \mathcal{L}^n \otimes \mathcal{O}_C(Q)) \\ &= \begin{cases} 1 & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now let $Z = X \times \mathbb{P}_k^{d-1}$ and $\mathcal{N} = \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}}(1)$. By the Kuenneth formula, we have that

$$\Gamma(Z, \mathcal{N}^n) = \Gamma(X, \mathcal{M}^n) \otimes_k \Gamma(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}}(n))$$

from which the conclusions of the example follow. \square

7. A GRADED FAMILY OF m -PRIMARY IDEALS WHICH DOES NOT HAVE A LIMIT

In this section we give an example showing lack of limits for families of m -primary ideals in non reduced rings. A family of ideals $\{I_n\}$ in a d -dimensional local ring R indexed by n is a family of ideals in R if $I_0 = R$ and $I_m I_n \subset I_{m+n}$ for all m, n . $\{I_n\}$ is an m_R -primary family if I_n is m_R -primary for $n \geq 1$. The limit

$$(28) \quad \lim_{n \rightarrow \infty} \frac{\ell(R/I_n)}{n^d}$$

is shown to exist in many cases in papers of Ein, Lazarsfeld and Smith [6], Mustata [15], Lazarsfeld and Mustata [13] and of the author [3]. It is shown in Theorem 5.9 [3] that the limit (28) always exists if $\{I_n\}$ is a graded family of m_R -primary ideals in a d -dimensional unramified equicharacteristic local ring R with perfect residue field.

Example 7.1. *Let k be a field and R be the non reduced d -dimensional local ring $R = k[[x_1, \dots, x_d, y]]/(y^2)$. There exists a graded family of m_R -primary ideals $\{I_n\}$ in R such that the limit*

$$\lim_{n \rightarrow \infty} \frac{\ell(R/I_n)}{n^d}$$

does not exist, even when n is constrained to lie in an arithmetic sequence. Here ℓ denotes the length of an R -module.

Proof. Let $\bar{x}_1, \dots, \bar{x}_d, \bar{y}$ be the classes of x_1, \dots, x_d, y in R . Let N_i be the set of monomials of degree i in the variables $\bar{x}_1, \dots, \bar{x}_d$. Let $\sigma(n)$ be the function defined in (18). Define m_R -primary ideals I_n in R by $I_n = (N_n, \bar{y}N_{n-\sigma(n)})$ for $n \geq 1$ (and $I_0 = R$).

We first verify that $\{I_n\}$ is a graded family of ideals, by showing that $I_m I_n \subset I_{m+n}$ for all $m, n > 0$. This follows since

$$I_m I_n = (N_{m+n}, \bar{y}N_{(m+n)-\sigma(m)}, \bar{y}N_{(m+n)-\sigma(n)})$$

and $\sigma(j) \leq \sigma(k)$ for $k \geq j$.

R/I_n has a k -basis consisting of

$$\{N_i | i < n\} \text{ and } \{\overline{y}N_j | j < n - \sigma(n)\}.$$

Thus

$$\ell(R/I_n) = \binom{n}{d} + \binom{n - \sigma(n)}{d}.$$

By an similar argument to that of the proof of Example 6.4, we obtain the conclusions of this example. \square

8. APPENDIX

In this appendix, we give a proof of Lemma 2.1 stated in Section 2. We begin with a proof of another lemma we will need.

Lemma 8.1. *Suppose that L is a graded linear series on a projective scheme X over a field k , and L is a finitely generated L_0 -algebra. Then*

$$(29) \quad q(L) = \begin{cases} \text{Krull dimension}(L) - 1 & \text{if Krull dimension}(L) > 0 \\ -\infty & \text{if Krull dimension}(L) = 0. \end{cases}$$

Proof. In the case when $L_0 = k$, the lemma follows from graded Noether normalization (Theorem 1.5.17 [2]). For a general graded linear series L , we always have that $k \subset L_0 \subset \Gamma(X, \mathcal{O}_X)$, which is a finite dimensional k -vector space since X is a projective k -scheme. L is thus a finitely generated k algebra. Let $m = \sigma(L)$ and $y_1, \dots, y_m \in L$ be homogeneous elements of positive degree which are algebraically independent over k . Extend to homogeneous elements of positive degree y_1, \dots, y_n which generate L as an L_0 -algebra. Let $B = k[y_1, \dots, y_n]$. We have that $\sigma(L) \leq \sigma(B) \leq \sigma(L)$ so $\sigma(B) = \sigma(L)$. By the first case ($L_0 = k$) proven above, we have that $q(B) = \text{Krull dimension}(B) - 1$ if $q(B) \geq 0$ and $q(B) = -\infty$ if $\text{Krull dimension}(B) = 0$. Since L is finite over B , we have that $\text{Krull dimension}(L) = \text{Krull dimension}(B)$. Thus the lemma holds. \square

8.1. Proof of Lemma 2.1. We will first establish the formulas (3) and (4). Since X is projective over k , we have an expression $X = \text{Proj}(A)$ where A is the quotient of a standard graded polynomial ring $R = k[x_0, \dots, x_n]$ by a homogeneous ideal I , which we can take to be saturated; that is, (x_0, \dots, x_n) is not an associated prime of I . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the associated primes of I . By graded prime avoidance (Lemma 1.5.10 [2]) there exists a form F in $k[x_0, \dots, x_n]$ of some positive degree c such that $F \notin \cup_{i=1}^t \mathfrak{p}_i$. Then F is a nonzero divisor on A , so that $A \xrightarrow{F} A(c)$ is 1-1. Sheafifying, we have an injection

$$(30) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(c).$$

Since $\mathcal{O}_X(c)$ is ample on X , there exists $f > 0$ such that $\mathcal{A} := \mathcal{L} \otimes \mathcal{O}_X(cf)$ is ample. From (30) we then have a 1-1 \mathcal{O}_X -module homomorphism $\mathcal{O}_X \rightarrow \mathcal{O}_X(cf)$, and a 1-1 \mathcal{O}_X -module homomorphism $\mathcal{L} \rightarrow \mathcal{A}$, which induces inclusions of graded k -algebras

$$L \subset \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \subset B := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{A}^n).$$

There exists a positive integer e such that \mathcal{A}^e is very ample on X . Thus, by Theorem II.5.19 and Exercise II.9.9 [8], $B' = \bigoplus_{n \geq 0} B_{en}$ is finite over a coordinate ring S of X and

$$(31) \quad \dim_k \Gamma(X, \mathcal{A}^{en}) = P_S(n)$$

for $n \gg 0$ where $P_S(n)$ is the Hilbert Polynomial of S .

Now B is a finitely generated module over B' . Thus

$$\text{Krull dimension}(B) = \text{Krull dimension}(B') = \text{Krull dimension}(S) = \dim X + 1 = d + 1.$$

Further, B is a finitely generated k -algebra so that $q(B) = \dim(X)$ by (29). Thus we have obtained formula (3),

$$q(L) \leq \dim X.$$

\mathcal{A} is ample on X , so that

$$\dim_k \Gamma(X, \mathcal{A}^n) = \chi(\mathcal{A}^n)$$

for $n \gg 0$. The Euler characteristic $\chi(\mathcal{A}^n)$ is a polynomial in n for $n \gg 0$ by Proposition 8.8a [9], which necessarily has degree d by (31), so there exists a positive constant γ such that

$$\dim_k L_n < \gamma n^d$$

for all n , giving us formula (4).

We will now establish formula (5). Suppose that $q(L) \geq 0$. Let L^i be the k -subalgebra of L generated by L_j for $j \leq i$. For i sufficiently large, we have that $q(L^i) = q(L)$. For such an i , since L^i is a finitely generated L_0 -algebra, we have that there exists a number e such that the Veronese algebra L^* defined by $L_n^* = (L^i)_{en}$ is generated as a L_0 -algebra in degree 1. Thus, since L_0 is an Artin ring, and L^* has Krull dimension $q(L) + 1$ by (29), L^* has a Hilbert polynomial $P(t)$ of degree $q(L)$, satisfying $\ell_{L_0}(L_n^*) = P(n)$ for $n \gg 0$ (Corollary to Theorem 13.2 [14]), where ℓ_{L_0} denotes length of an L_0 module, and thus $\dim_k L_n^* = (\dim_k L_0)P(n)$ for $n \gg 0$. Thus there exists a positive constant α such that $\dim_k L_n^* > \alpha n^{q(L)}$ for all n , and so

$$\dim_k L_{en} > \alpha n^{q(L)}$$

for all positive integers n , which is formula (5).

Finally, we will establish the fourth statement of the Lemma. Suppose that X is reduced and $0 \neq L_n$ for some $n > 0$. Consider the graded k -algebra homomorphism $\varphi : k[t] \rightarrow L$ defined by $\varphi(t) = z$ where $k[t]$ is graded by giving t the weight n . The kernel of φ is weighted homogeneous, so it is either 0 or (t^s) for some $s > 1$. Thus if φ is not 1-1 then there exists $s > 1$ such that $z^s = 0$ in L_{ns} . We will show that this cannot happen. Since z is a nonzero global section of $\Gamma(X, \mathcal{L}^n)$, there exists $Q \in X$ such that the image of z in \mathcal{L}_Q^n is σf where $f \in \mathcal{O}_{X,Q}$ is nonzero and σ is a local generator of \mathcal{L}_Q^n . The image of z^s in $\mathcal{L}_Q^{sn} = \sigma^s \mathcal{O}_{X,Q}$ is $\sigma^s f^s$. We have that $f^s \neq 0$ since $\mathcal{O}_{X,Q}$ is reduced. Thus $z^s \neq 0$. We thus have that φ is 1-1, so $q(L) \geq 0$.

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